On Quantum de Finetti Theorems

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Overview

de Finetti theorems

- classical & quantum exchangeability
- known de Finetti theorems

permutation-invariant states

- approximation by product states
- generalisation: highest-weight vectors, coherent states

symmetric Werner states

- Schur- & shifted Schur functions, lower bounds

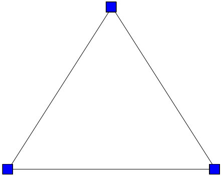
joint work with M. Christandl, G. Mitchison and R. Renner

quant-ph/0602130

Symmetric States: Motivation

Hamiltonian $H = \sum_{(i,j)\in E} h_{ij}$ on n particles

$h_{ij} = (h)_{ij} \otimes \mathrm{id}_{V\setminus\{i,j\}}$



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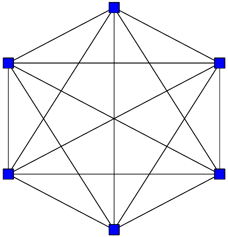

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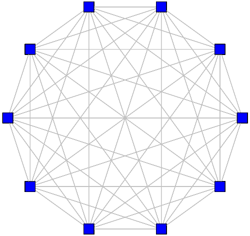

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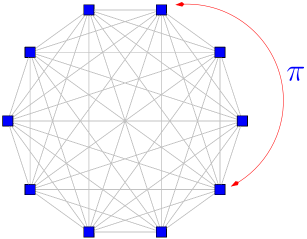

- ground state $|\Psi\rangle$ is permutation-invariant:
  $\pi|\Psi\rangle = |\Psi\rangle$ for all $\pi \in S_n$

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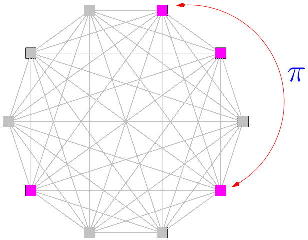


- ground state $|\Psi\rangle$ is permutation-invariant:

$$\pi|\Psi\rangle = |\Psi\rangle \text{ for all } \pi \in S_n$$

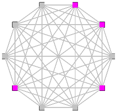
- reduced k-particle density matrix $\rho^k = \mathrm{tr}_{n-k}|\Psi\rangle\langle\Psi|$ is permutation-invariant:

$$\boxed{\pi\rho^k\pi^\dagger = \rho^k} \text{ for all } \pi \in S_n$$

Quantum & Classical Exchangeability

Definition: ρ^k on $(\mathbb{C}^d)^{\otimes k}$ is *n-exchangeable* if $\exists \rho^n$ on $(\mathbb{C}^d)^{\otimes n}$ s. th.

$$\rho^k = \mathrm{tr}_{n-k}(\rho^n) \qquad \text{and} \qquad \rho^n = \pi \rho^n \pi^\dagger \qquad \text{for all } \pi \in S_n$$



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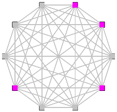

Example: $\rho^k = \int d\mu(\sigma)\sigma^{\otimes k}$ is n-exchangeable for $n \geq k$:

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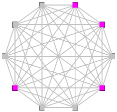

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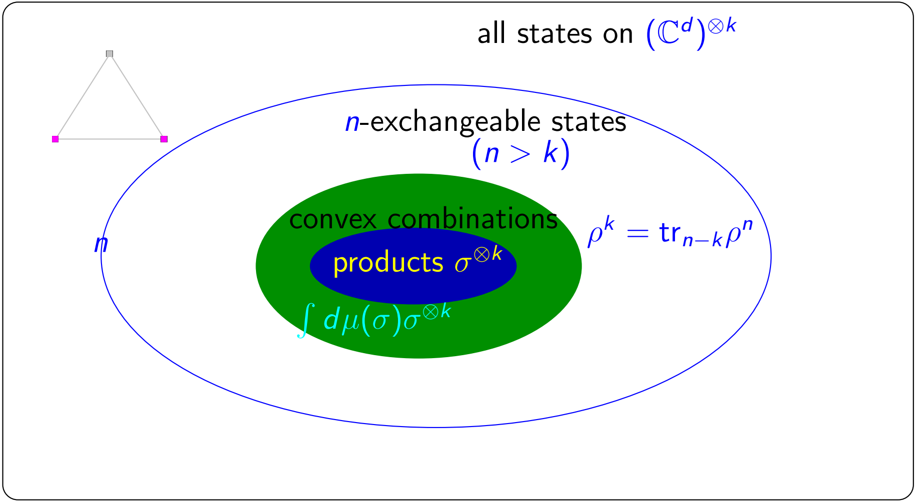
Definition: $P_{X_1 \cdots X_k}$ is n-*exchangeable* if $\exists P_{X_1 \cdots X_n}$ s. th.

$$P_{X_1 \cdots X_n}(x_1, \ldots, x_n) = P_{X_1 \cdots X_n}(x_{\pi(1)}, \ldots, x_{\pi(n)})$$

for all $\pi \in S_n$

de Finetti theorems



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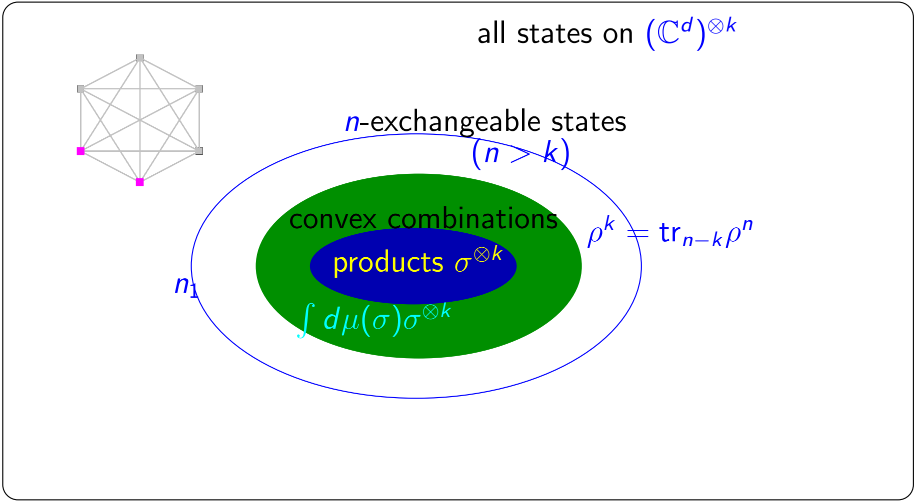
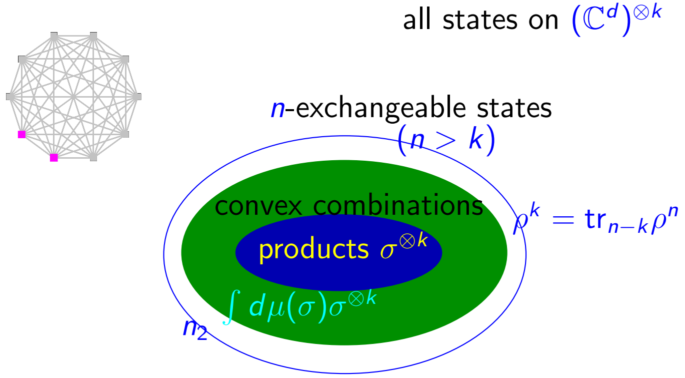
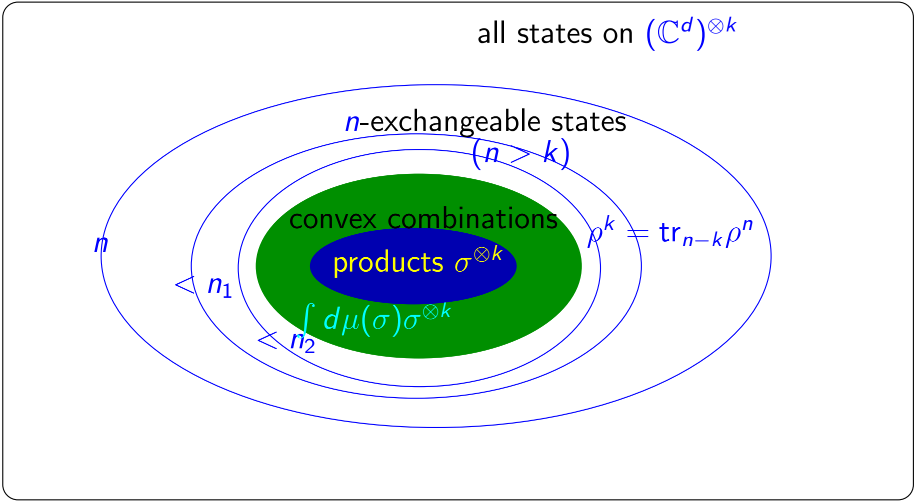



all states on $(\mathbb{C}^d)^{\otimes k}$
n-exchangeable states
$(n > k)$
convex combinations
products $\sigma^{\otimes k}$
$\int d\mu(\sigma)\sigma^{\otimes k}$
n_2
$\rho^k = \mathrm{tr}_{n-k}\rho^n$

de Finetti theorems



Various de Finetti Theorems

Classical: $P_{X_1 \cdots X_k}$ distribution of d-ary R.V.

- $\equiv \int d\mu(Q)\ Q^k$ if ∞-exchangeable [deFin37]
- $\approx \int d\mu(Q)\ Q^k$ if n-exchangeable [DiaFre80]

error $\min\{d \cdot \frac{k}{n}, \frac{k(k-1)}{n}\}$ (optimal!) d: alphabet size

Quantum: ρ^k state on $(\mathbb{C}^d)^{\otimes k}$

- $\equiv \int d\mu(\sigma)\ \sigma^{\otimes k}$ if ∞-exchangeable

[Stø69, HuMo76,RaWe89,Pe90,CaFuSc02]

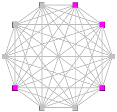
- $\approx \int d\mu(\sigma)\ \sigma^{\otimes k}$ if n-exchangeable

error $d^2 \cdot \frac{k}{n}$ d: dimension

New de Finetti theorems

(for **mixed** symmetric states)

Given: $\rho^k = \mathrm{tr}_{n-k}\rho^n$ with ρ^n on $(\mathbb{C}^d)^{\otimes n}$ symmetric



$\exists$ probability measure μ such that

$$\|\rho^k - \int d\mu(\sigma)\ \sigma^{\otimes k}\| \leq d^2 \cdot \frac{k}{n}\ .$$

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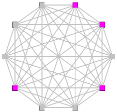

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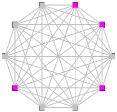
dependence on k, n tight for fixed d!

New de Finetti theorems

(for **pure** symmetric states)

Given: $\rho^k = \mathrm{tr}_{n-k}|\Psi\rangle\langle\Psi|$ with $|\Psi\rangle \in (\mathbb{C}^d)^{\otimes n}$ symmetric



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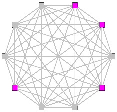

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$$\|\rho^k - \int d\mu(\varphi)\ |\varphi\rangle\langle\varphi|^{\otimes k}\| \leq d \cdot \frac{k}{n} \ .$$

Observation: $|\varphi\rangle^{\otimes k} = U^{\otimes k}|0\rangle^{\otimes k}$ for $U \in \mathcal{U}(d)$ (unitary group)

Group-theoretic observations

The unitary group $\mathcal{U}(d)$ acts on $\mathcal{H}^{\otimes n}$ by ($\mathcal{H} \cong \mathbb{C}^d$)

$$U|\Psi\rangle := U^{\otimes n}|\Psi\rangle$$

Properties:

- action irreducible on the symmetric subspace $\mathrm{Sym}^n(\mathcal{H}) \subset \mathcal{H}^{\otimes n} \Leftrightarrow$ representation $U_{(n)} = U_{\square\square\square\square}$

- $|0\rangle^{\otimes n} \in \mathrm{Sym}^n(\mathcal{H})$ is highest weight vector:
$$U_{(n)}(\mathrm{diag}[h_1, \ldots, h_d])|0\rangle^{\otimes n} = h_1^n|0\rangle^{\otimes n}$$

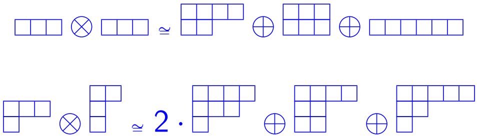
- product property:
$$\begin{array}{ccccc} \mathrm{Sym}^n(\mathcal{H}) & \subset & \mathrm{Sym}^k(\mathcal{H}) & \otimes & \mathrm{Sym}^{n-k}(\mathcal{H}) \\ |0\rangle^{\otimes n} & = & |0\rangle^{\otimes k} & \otimes & |0\rangle^{\otimes n-k} \end{array}$$

Generalising to arbitrary representations $U_{\mu+\nu}$ of $\mathcal{U}(d)$

Clebsch-Gordan-Expansion:

$$U_\mu \otimes U_\nu \cong \bigoplus_\lambda c^\lambda_{\mu\nu} U_\lambda$$

Examples (d=3):



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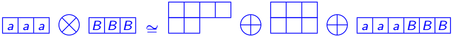

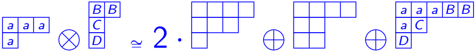

general observation:

$$U_\mu \otimes U_\nu \supset U_{\mu+\nu},$$

product property of highest weight vector: $|v_\mu\rangle \otimes |v_\nu\rangle = |v_{\mu+\nu}\rangle$

Highest Weight Theorem

Notation

- U_μ: irreducible representation of $\mathcal{U}(d)$ with highest weight μ
- $|v_\mu\rangle$ highest weight vector in U_μ

Re-interpreting....

	standard	highest weight-version
pure states $\|\Psi\rangle \in$	$\mathrm{Sym}^n(\mathcal{H})$	$U_{\mu+\nu} \subset U_\mu \otimes U_\nu$
partial trace	tr_{n-k}	tr_ν
approximation by	$U^{\otimes k}\|0\rangle^{\otimes k}$	$U_\mu(U)\|v_\mu\rangle$
error given by dim of	$\mathrm{Sym}^k(\mathcal{H})$, $\mathrm{Sym}^{n-k}(\mathcal{H})$	U_μ, U_ν

permutation-invariance $\pi\rho\pi^\dagger = \rho$

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$+$

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$=$ **Symmetric Werner States**

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Why stronger assumptions?

- fewer parameters $\Rightarrow$ different techniques
- rich class of examples $\Rightarrow$ lower bounds
- ...

Symmetric Werner states

states ρ^n on $(\mathbb{C}^d)^{\otimes n}$ with S_n- and $\mathcal{U}(d)$-invariance

$$\pi \rho^n \pi^\dagger = \rho^n \text{ for all } \pi \in S_n$$

and

$$U^{\otimes n} \rho^n (U^{\otimes n})^\dagger = \rho^n \text{ for all } U \in \mathcal{U}(d)$$

compact parametrisation by distribution $\{\alpha_\lambda\}_\lambda$

$$\boxed{\rho = \sum_\lambda \alpha_\lambda \rho_\lambda}$$

where each ρ_λ is (normalised) projector onto $U_\lambda \otimes V_\lambda$ in

$$(\mathbb{C}^d)^{\otimes n} \cong \bigoplus_\lambda U_\lambda \otimes V_\lambda \qquad \text{(Schur-Weyl-duality)}$$

Werner states form a convex polytope

Werner- and "Product"-Werner-states

example (extremal points)

$d = 2, n = 2$: $\rho_{\yng(2)}$ triplet $\rho_{\yng(1,1)}$ singlet

$d = 2, n = 4$: $\rho_{\yng(4)}$, $\rho_{\yng(3,1)}$, $\rho_{\yng(2,2)}$

"Twirling":

$$\boxed{\mathbb{T}(\rho) := \int dU \; U^{\otimes n} \rho (U^{\dagger})^{\otimes n}}$$

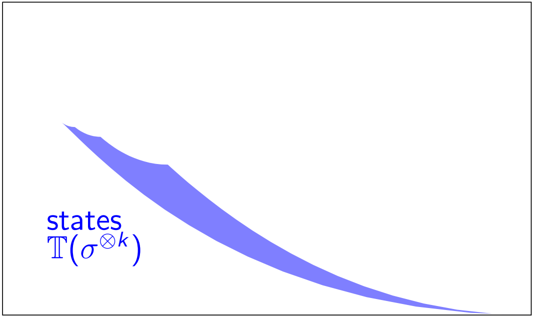
defines Werner state for every ρ on $(\mathbb{C}^d)^{\otimes n}$

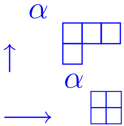
"Product" Werner states:

$$\mathbb{T}(\sigma^{\otimes n})$$

is convex combination of products, depends only on spectrum of σ

Example: Symmetric Werner States with $d = 5, k = 4$





twirled product states

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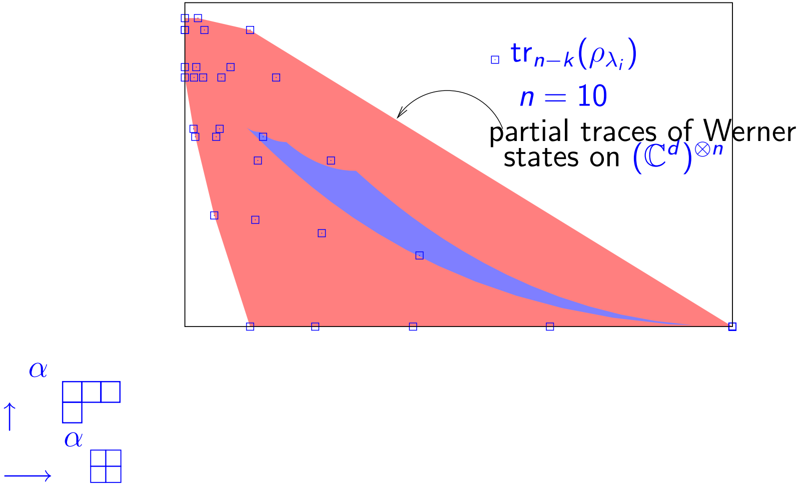

points: partial traces of symmetric Werner states on $n = 10$ systems

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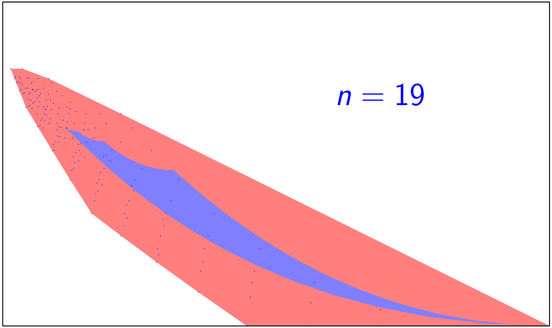

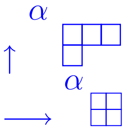

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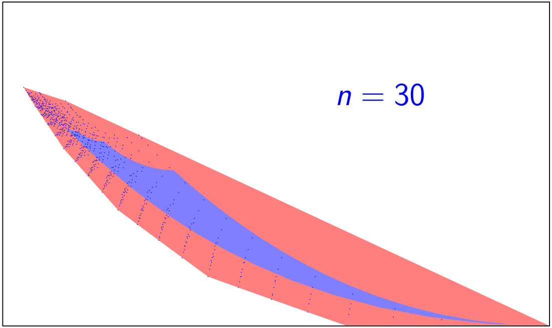

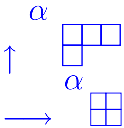

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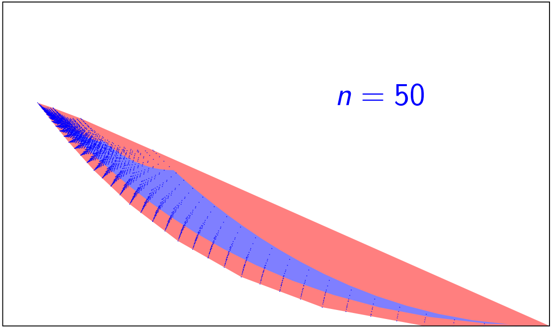

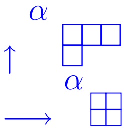

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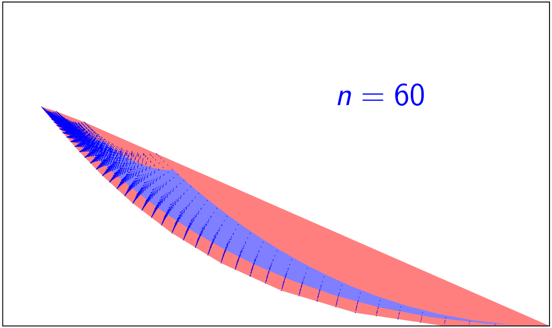

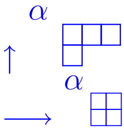

points: partial traces of symmetric Werner states on $n = 60$ systems

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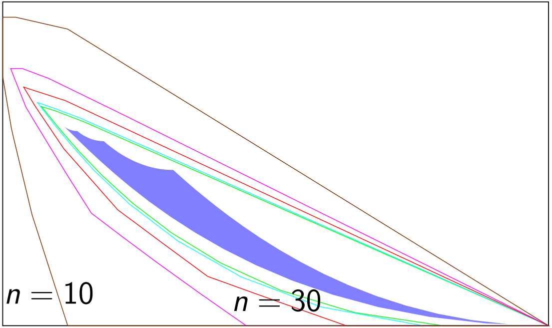
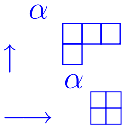


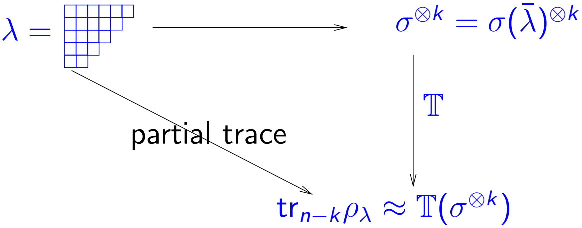


observation: partial traces $\mathrm{tr}_{n-k}(\rho_\lambda)$ are close to having form $\mathbb{T}(\sigma^{\otimes k})$

Observations:

- partial traces $\mathrm{tr}_{n-k}(\rho_\lambda)$ are close to having form $\mathbb{T}(\sigma^{\otimes k})$
- the map $\lambda \mapsto \sigma(\bar{\lambda}) := \mathrm{diag}(\bar{\lambda})$, where $\bar{\lambda} := (\frac{\lambda_1}{n}, \ldots, \frac{\lambda_d}{n})$ associates states/spectra to YD λ

de Finetti-Theorem: diagram is “approximately commutative”



Partial traces of Werner States

formulas for Werner states $\sum_\mu \alpha_\mu \rho_\mu$

state	type of state	coefficient $\frac{\alpha_\mu}{\dim V_\mu}$	function
$\mathbb{T}(\mathrm{diag}(r)^{\otimes k})$	twirled product state	$s_\mu(r)$	Schur
$\mathrm{tr}_{n-k}(\rho_\lambda)$	traced-out sym. W.	$s^\star_\mu(\lambda)/(n \downharpoonleft k)$	shifted Schur

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Proof of de Finetti for Werner States(n large):

use $s^\star_\mu(n\bar{\lambda})/(n \downharpoonleft k) \approx s_\mu(\bar{\lambda})$

Lower bounds

What's the "best possible" quantum de Finetti theorem?

Lower bounds

Known result for classical n-exchangeable distributions [DiaFre80]

$\exists\ P_{X_1 \cdots X_k}$ n-exchangeable with X_i binary,

$$\| P_{X_1 \cdots X_k} - \int d\mu(Q)\ Q^k \| \geq c \cdot \frac{k}{n} \qquad \forall \mu$$

Consequences:

- $\frac{k}{n}$ necessary in any quantum de Finetti theorem.
- d-dependence?

A lower bound on the d-dependence

Claim: $\exists$ n-exchangeable state ρ on $(\mathbb{C}^d)^{\otimes 2}$ s.th.

$$\|\rho - \int d\mu(\sigma)\, \sigma^{\otimes 2}\| \geq \frac{d}{n} \qquad \forall \mu$$

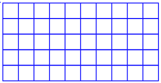
Dimension-independent de Finetti-theorems impossible!

A lower bound on the d-dependence

Claim: $\exists$ n-exchangeable state ρ on $(\mathbb{C}^d)^{\otimes 2}$ s.th.

$$\|\rho - \int d\mu(\sigma)\ \sigma^{\otimes 2}\| \geq \frac{d}{n} \qquad\qquad \forall \mu$$

Proof: take $\rho = \mathrm{tr}_{n-2}\rho_\lambda$, with

$\lambda =$  (d rows)

- compute $\alpha_{\substack{\square\\\square}}$ for this state
- show that $\alpha_{\substack{\square\\\square}} \leq \frac{d-1}{2d}$ for all states of the form $\int d\mu(\sigma)\sigma^{\otimes 2}$

Open Problems

- lower bounds: is $d \cdot \frac{k}{n}$ optimal?

- other symmetry types/assumptions: e.g.,
  - de Finetti theorem for channels/non-local systems?
  - exponentially small error with "almost" product states

work in progress