# Classification of topologically protected gates for local stabilizer codes

Robert König

joint work with

Sergey Bravyi

# T-W JCQC

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Tsinghua University Institute for Interdisciplinary Information Sciences

IQC Institute for Quantum Computing

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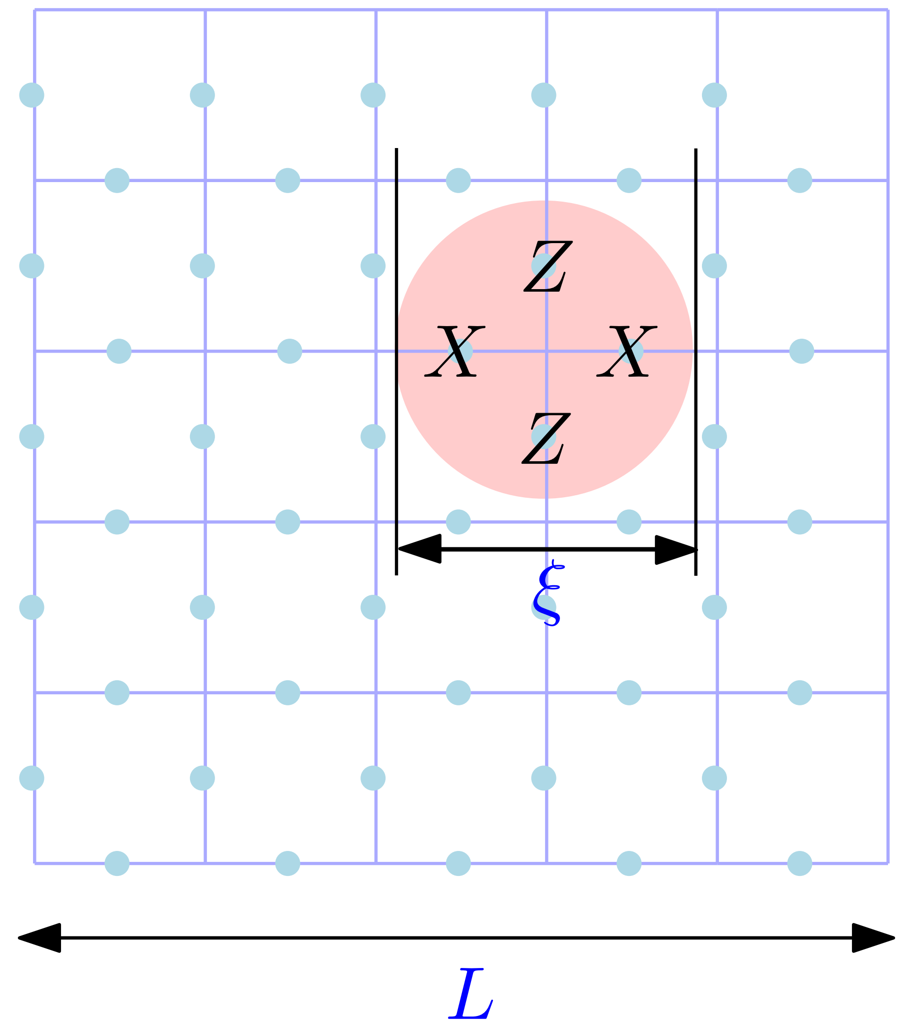
# Local (topological) stabilizer codes

$D$-dimensional array of qubits
of size $L$

*local* stabilizer generators:
support of any generator
has diameter $\xi = O(1)$

code distance $d \gg \xi$

# of encoded qubits $k$

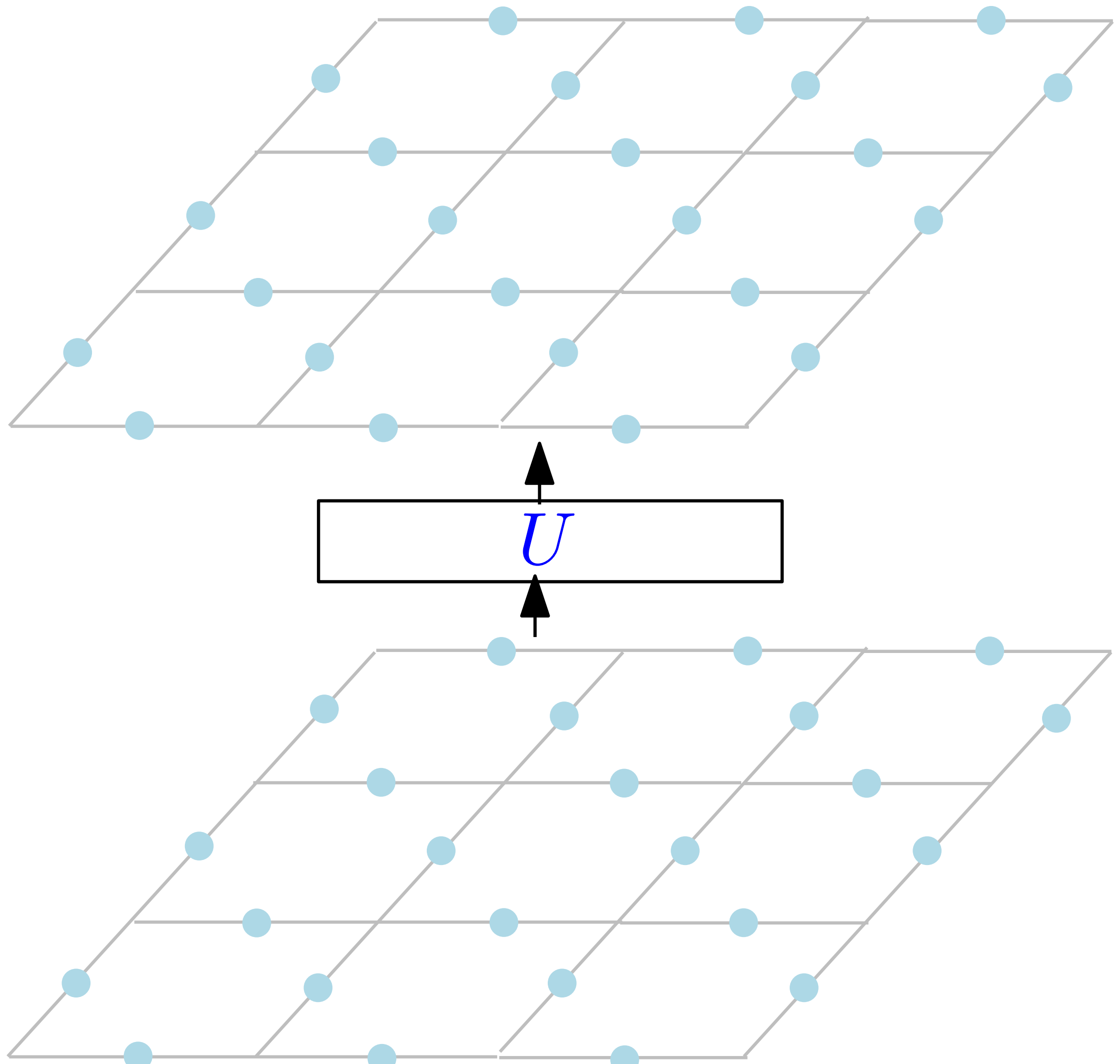



examples:

- toric code/surface codes [Kitaev'97, Bravyi, Kitaev'98]
- color codes [Bombin, Martin-Delgado'06]
- 3D self-correcting memories [Haah 12] and [Michnickis 12]
- surface code with twists [Bombin'10]
- ...

# Protected gates?

**logical gate**: unitary $U$ preserving codespace $\mathcal{L}$: $U\mathcal{L} = \mathcal{L}$




fault-tolerance properties depend structure of $U$

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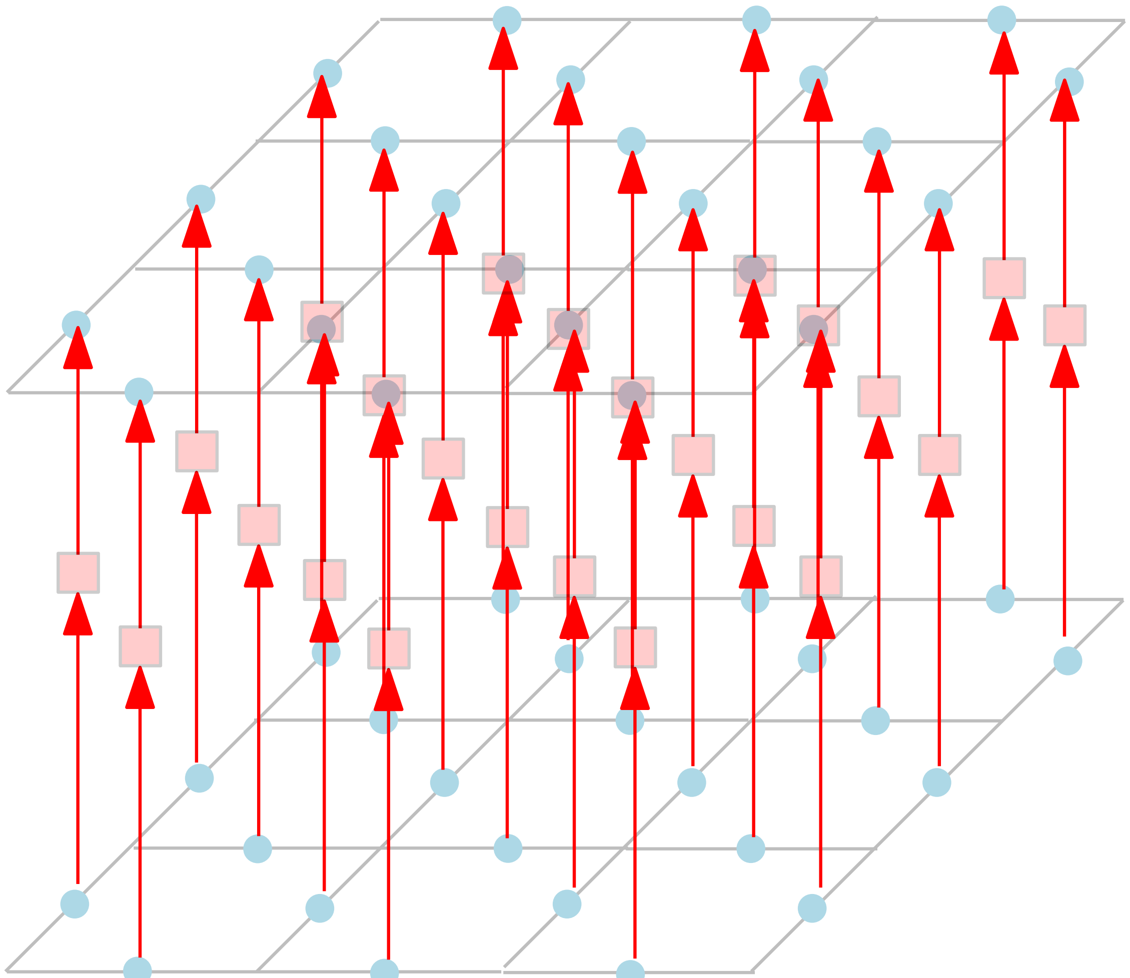


fault-tolerance properties depend structure of $U$

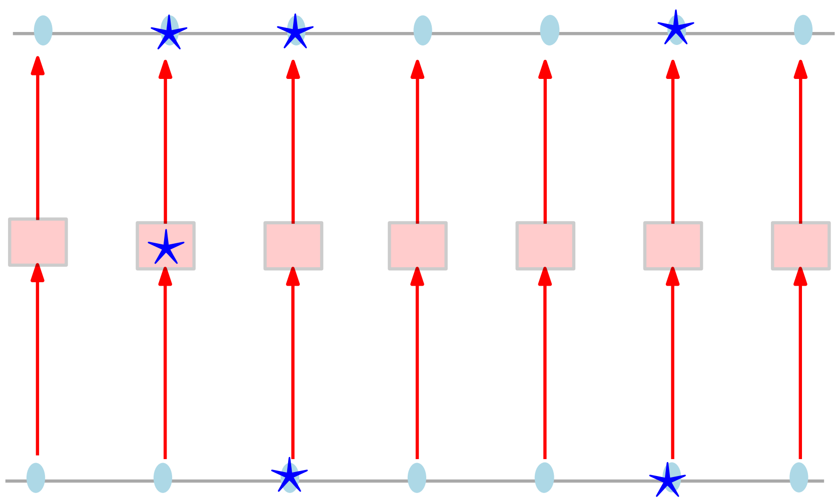
example of a protected gate:

**transversal gate**

# Protected gates?

**logical gate**: unitary $U$ preserving codespace $\mathcal{L}$: $U\mathcal{L} = \mathcal{L}$

$\star$ error locations



fault-tolerance properties depend structure of $U$

example of a protected gate:

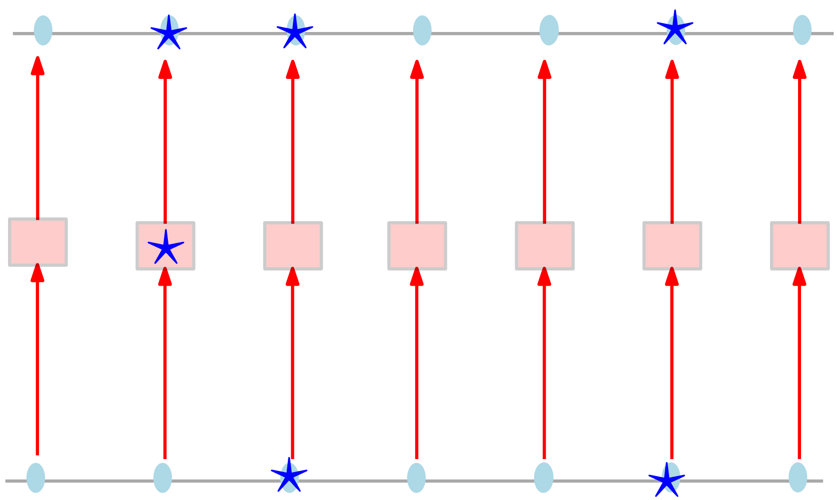
**transversal gate**

- preexisting errors do not spread
- faulty unitaries only introduce local errors

*when applying a transversal gate.*

# Limitations on transversal encoded gates

⋆ error locations



*General (non-stabilizer) codes:*

Theorem: Transversal encoded gates generate a **finite group**.

[Eastin, Knill '09]

*Proof uses theory of Lie groups.*

*2D surface codes:*

Theorem: Suppose the stabilizer group has no generators of weight $2$. Then all transversal gates are in the **Clifford group**.

[Sarvepalli, Raussendorf '09]

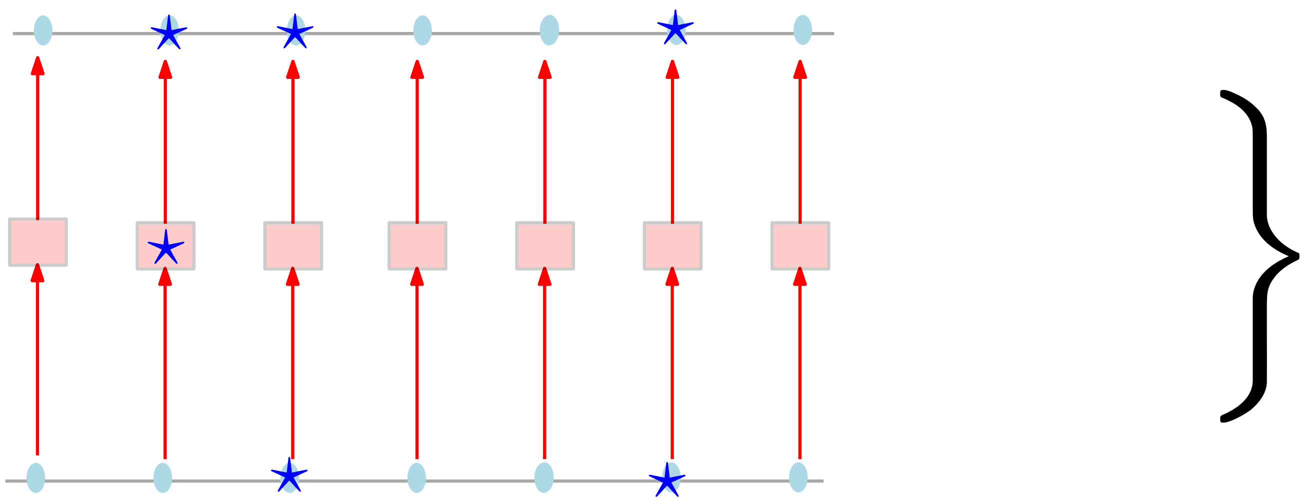
*Proof uses theory of matroids.*

# A more general notion of protected gates?

⋆ error locations

**transversal gate**

≡

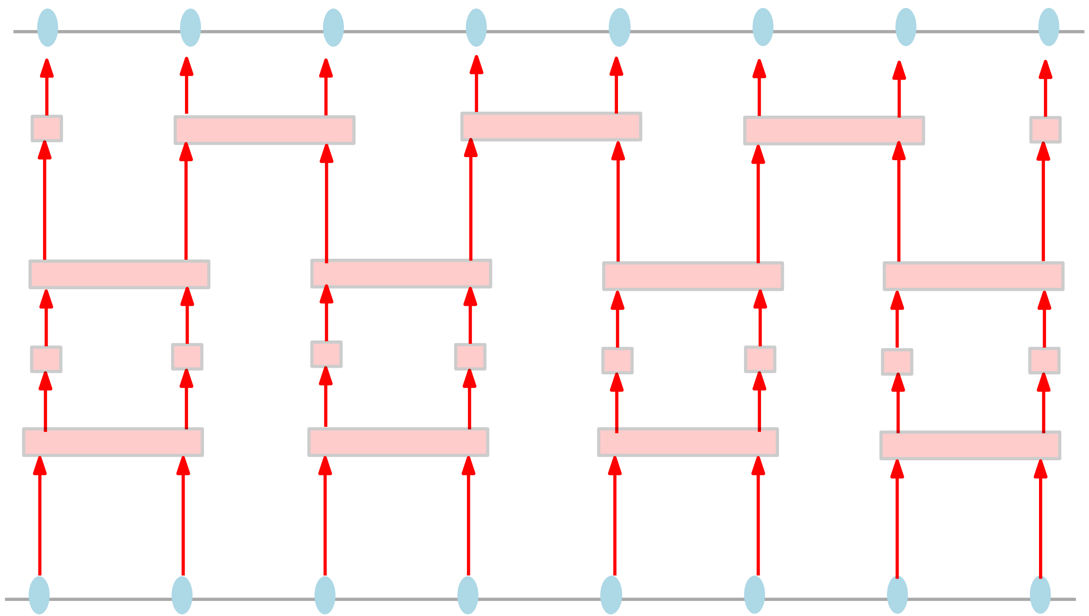


depth-1 quantum circuit

- preexisting errors do not spread
- faulty unitaries only introduce local errors

*when applying a transversal gate.*

# A definition of protected gates



**protected gate**

$\equiv$

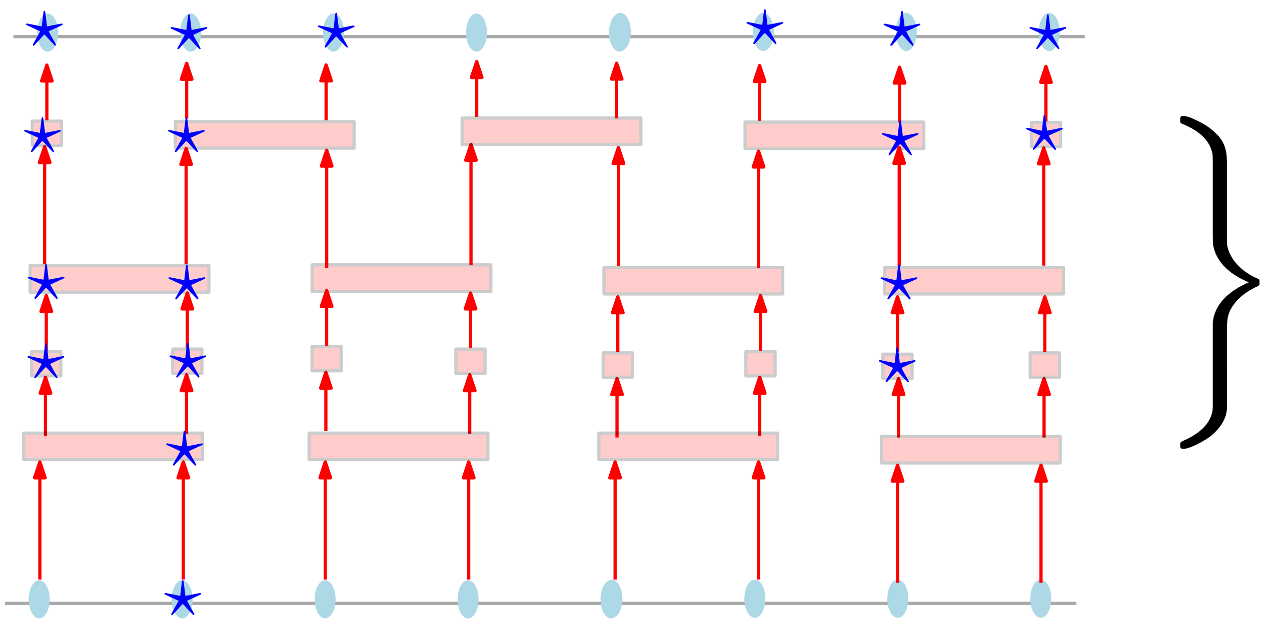
implementable by

constant-depth quantum circuit

- preexisting errors only spread to a *constant-width causal cone*
- faulty unitaries introduce errors restricted to *causal cone*

*when applying a gate realized by a* **constant-depth circuit**

# A definition of protected gates

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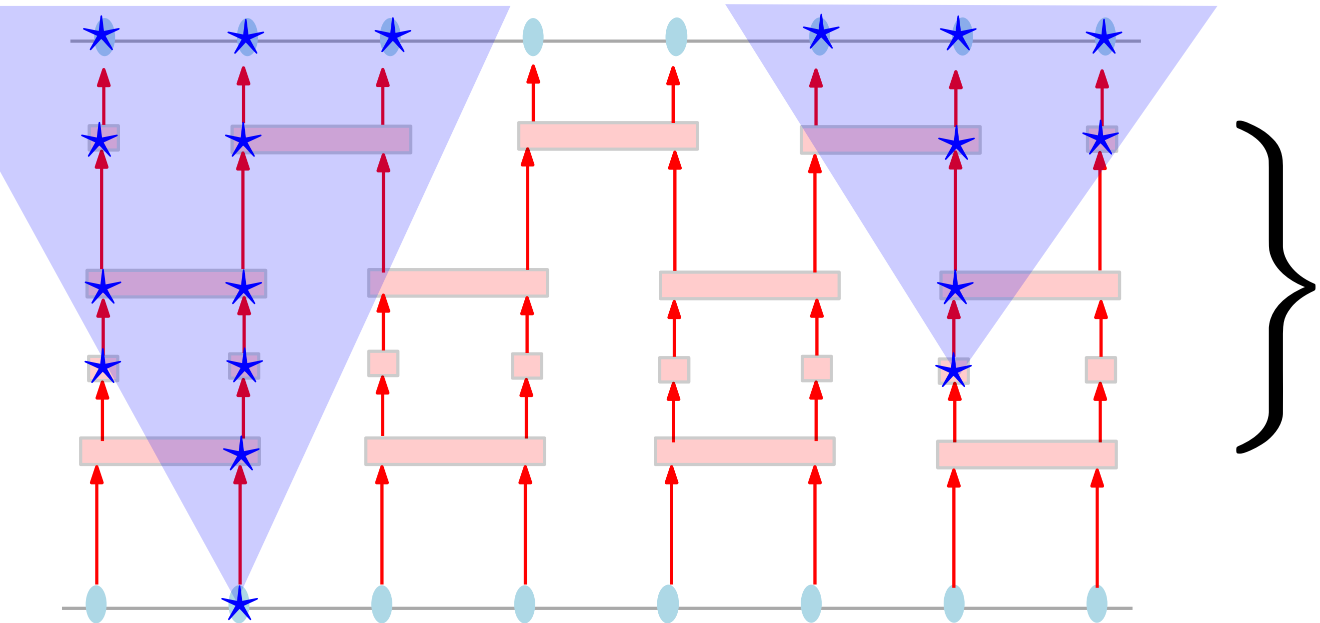
# A definition of protected gates




**protected gate**

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# The Clifford hierarchy and local stabilizer codes

Level 1: Pauli group [Gottesman, Chuang '99]

Level 2: Clifford group

Level 3: $\pi/8$-gate, Toffoli gate, $\Lambda(S)$, etc.

Level $j+1$: $\mathcal{C}_{j+1} = \{\bar{U} \in \mathsf{U}(2^k) \mid \bar{U}\mathcal{C}_1\bar{U}^\dagger \subseteq \mathcal{C}_j\}$

$\mathcal{C}_1$: Pauli group; $\mathcal{C}_j$: Level $j$

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($\mathcal{C}_1$: Pauli group; $\mathcal{C}_j$: Level $j$)

Properties:

- $\mathcal{C}_1 \subset \mathcal{C}_2 \subset \cdots \subset \mathcal{C}_j \subset \mathcal{C}_{j+1} \subset \cdots$
- $\begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^j} \end{pmatrix} \in \mathcal{C}_j \backslash \mathcal{C}_{j-1}$

# The Clifford hierarchy and local stabilizer codes

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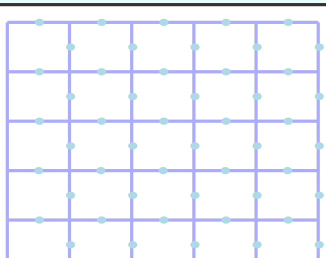
($\mathcal{C}_1$: Pauli group; $\mathcal{C}_j$: Level $j$)

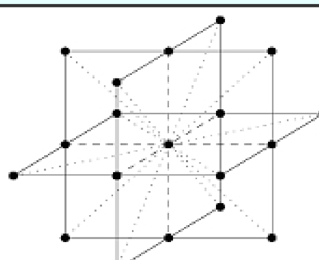
**Theorem:** For a $D$-dimensional local stabilizer code: $(D \geq 2)$

encoded gates implementable by a constant-depth circuit

belong to the level $D$ of the Clifford hierarchy.



Clifford group $\mathcal{C}_2$

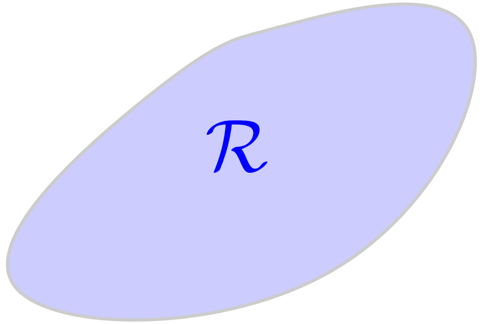


$\mathcal{C}_3$

# Proof tool I: the union lemma

Def: $\mathcal{R}$ correctable region $:\Leftrightarrow$ any logical Pauli operator supported on $\mathcal{R}$ acts as identity on code space

Example: number of qubits $|\mathcal{R}| < d$



# Proof tool I: the union lemma

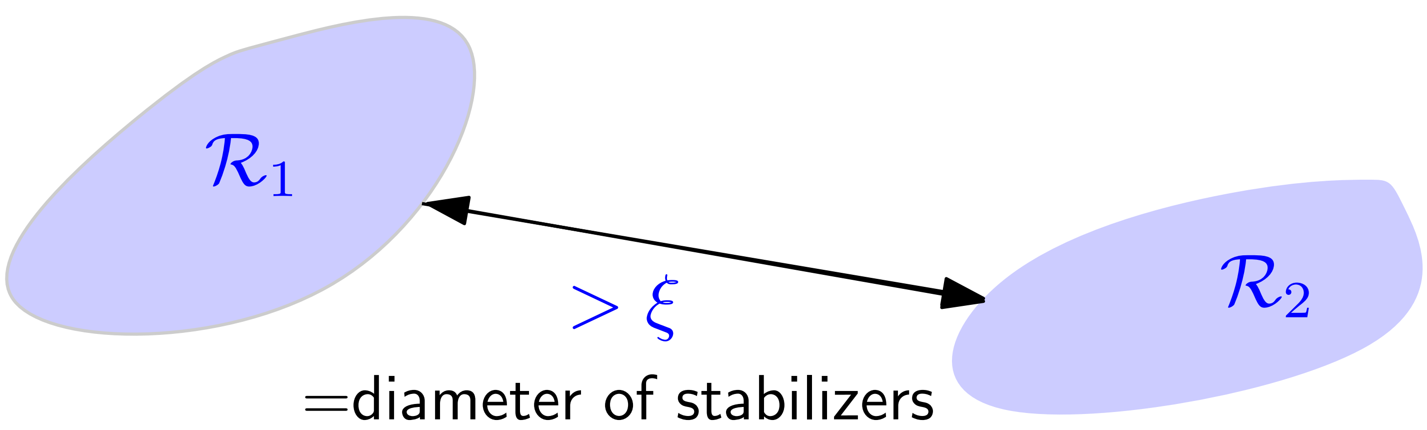
**Def:** $\mathcal{R}$ correctable region $:\Leftrightarrow$ any logical Pauli operator supported on $\mathcal{R}$ acts as identity on code space

**Union lemma:** [Bravyi, Poulin, Terhal '10] [Haah, Preskill'10]

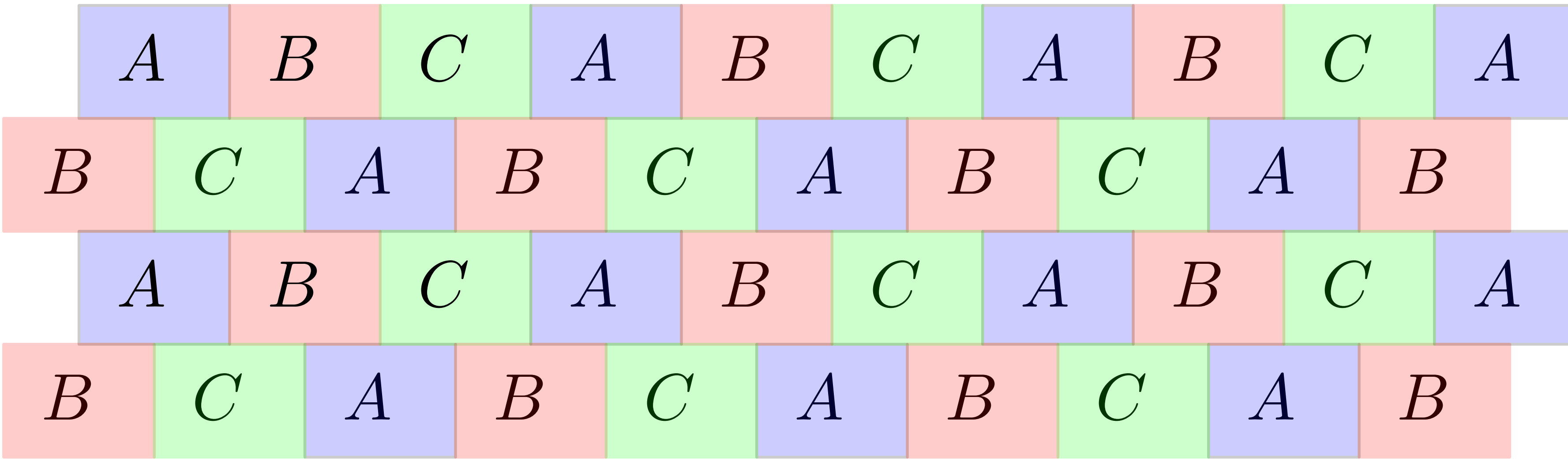
$\mathcal{R}_1, \mathcal{R}_2$ correctable regions,

distance$(\mathcal{R}_1, \mathcal{R}_2) > \xi$

$\Rightarrow$ $\mathcal{R}_1 \cup \mathcal{R}_2$ correctable



# Application of union Lemma: partition of lattice




in $D = 2$: 3 disjoint correctable regions $A, B, C$

by application of the union Lemma

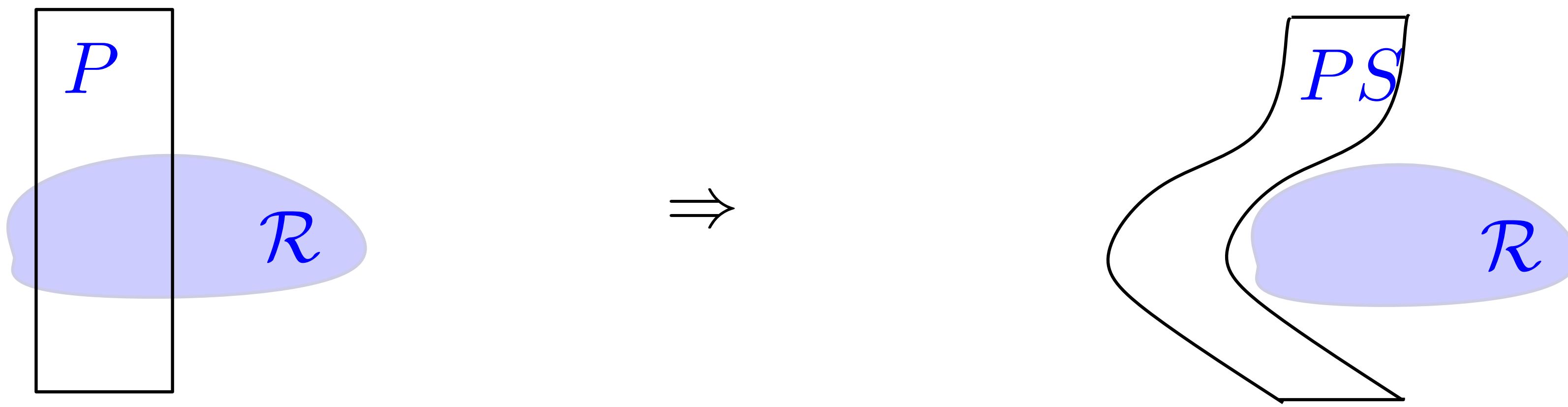
(in $D$: $D + 1$ disjoint correctable regions)

# Proof tool II: the cleaning lemma

**Def:** $\mathcal{R}$ correctable region $:\Leftrightarrow$ any logical Pauli operator supported on $\mathcal{R}$ acts as identity on code space

**Cleaning lemma:** [Bravyi, Terhal '08]

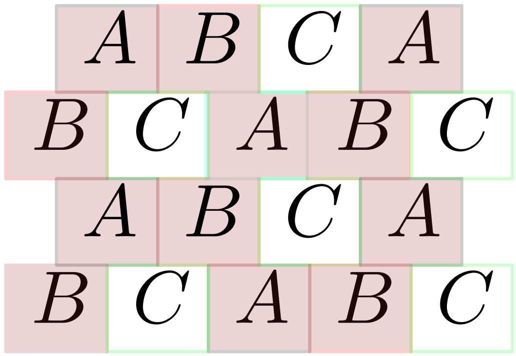
$\mathcal{R}$ correctable region, $P$ logical Pauli operator $\Rightarrow$ $\exists$ stabilizer $S$ such that $PS$ is supported *outside* $\mathcal{R}$



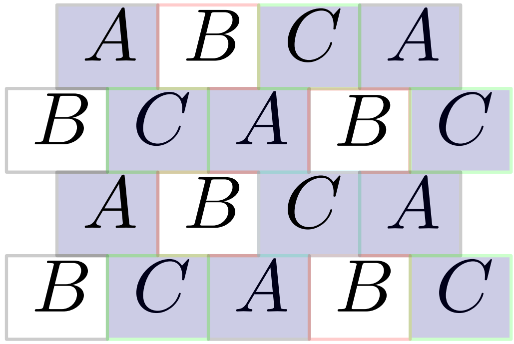

support of S: contained in $\xi$-neighborhood of $\mathcal{R}$

# Application of cleaning lemma

$P =$




$Q =$




(arbitrary)
logical Pauli operators
after cleaning

# Application of cleaning lemma

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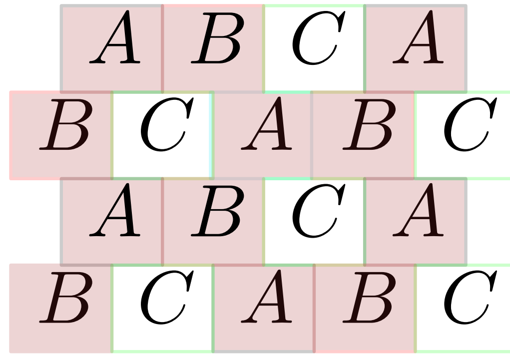


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logical Pauli operators
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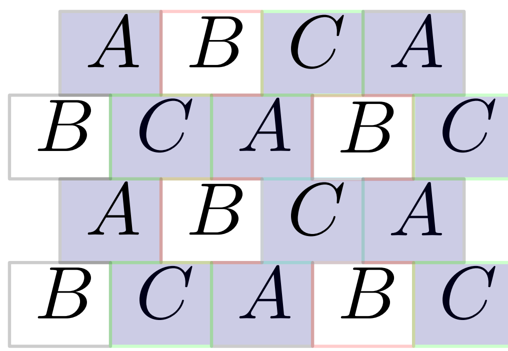
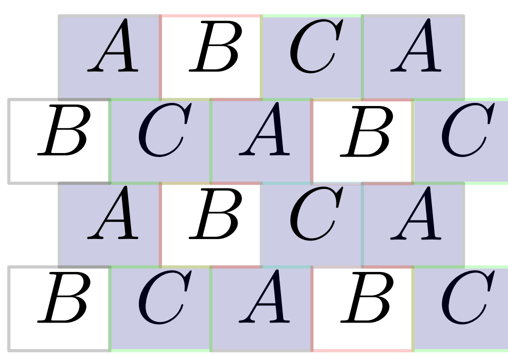



application of
a **transversal gate** $U$

(constant-depth: similar)

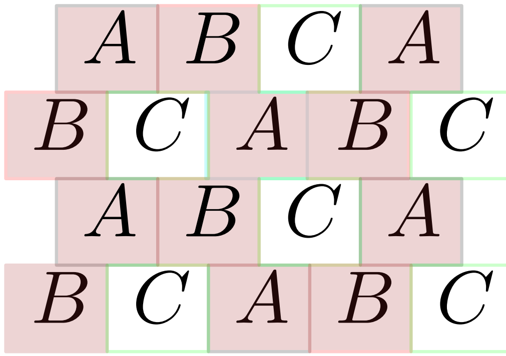
$Q_U =$
$UQU^\dagger =$




($Q_U$ is also transversal!)

# Application of cleaning lemma

$P =$

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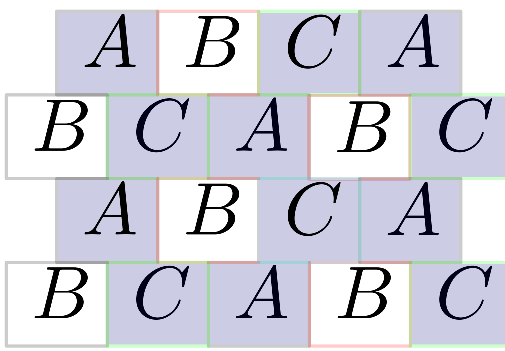


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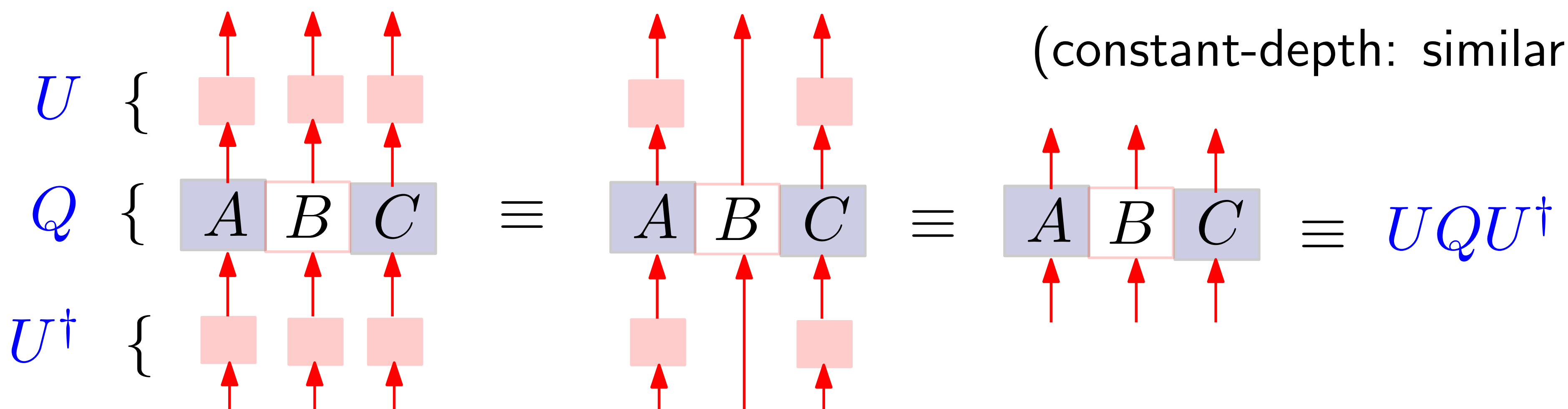
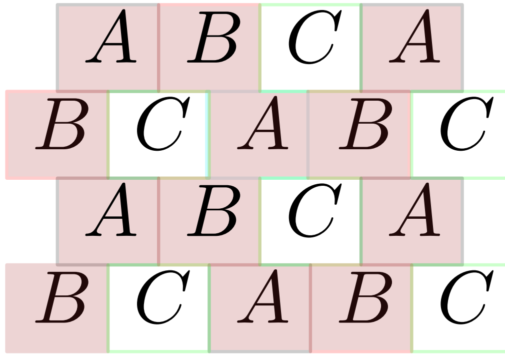


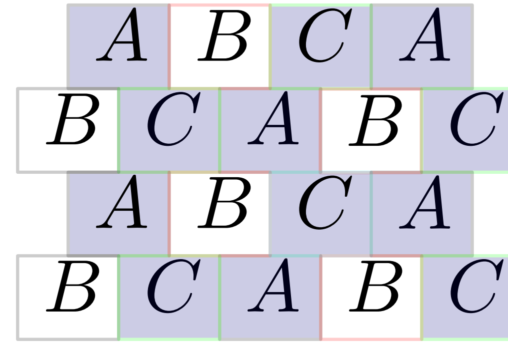
# Transversal gate $U$ & support of 'group commutator'

$P =$




Pauli

$Q_U = UQU^\dagger =$




transversal

**Claim:** $U\big|_{\mathcal{L}}$ is an encoded Clifford group element

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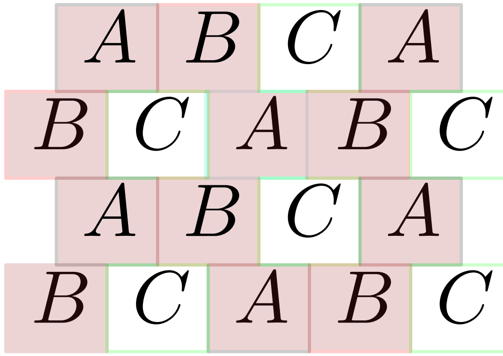


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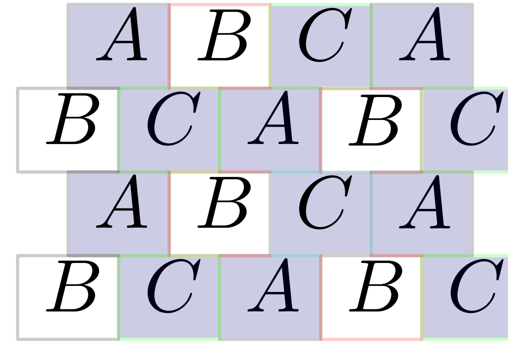
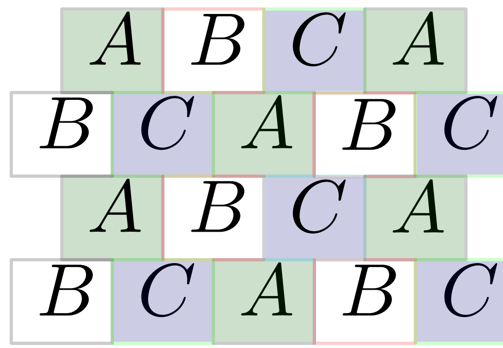



transversal

$PQ_U P^\dagger =$

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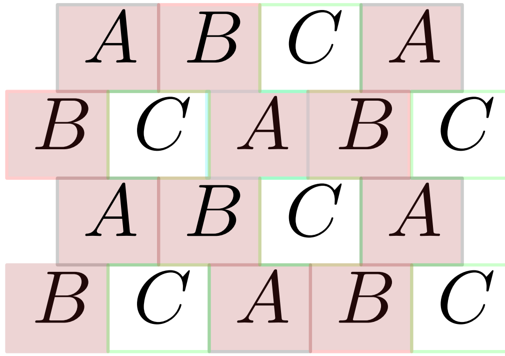


Pauli

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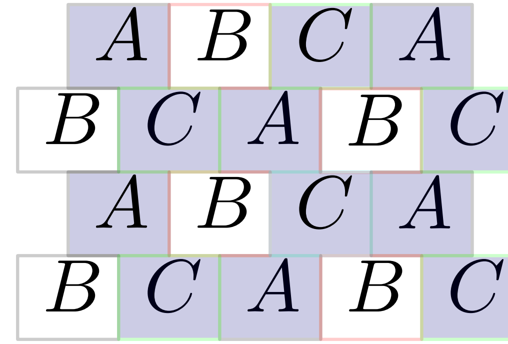
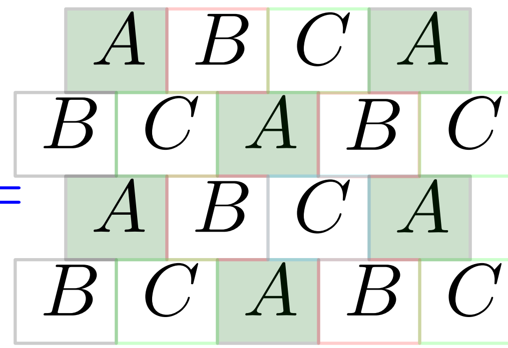



transversal

$PQ_UP^\dagger Q_U^\dagger =$

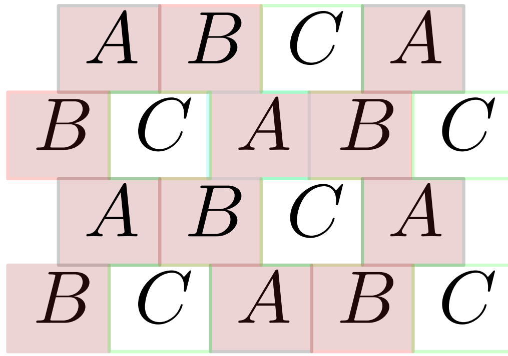



supported on correctable region $A$

**Claim:** $U\big|_{\mathcal{L}}$ is an encoded Clifford group element

# Transversal gate $U$ & support of 'group commutator'

$P =$




Pauli

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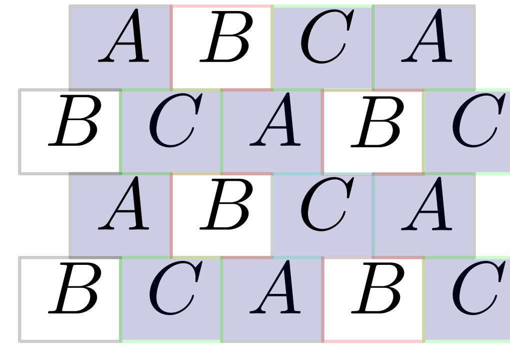
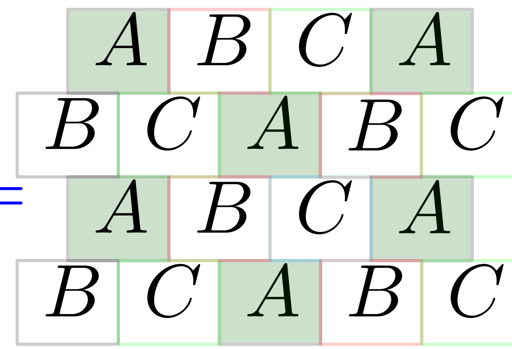



transversal

$PQ_U P^\dagger Q_U^\dagger =$



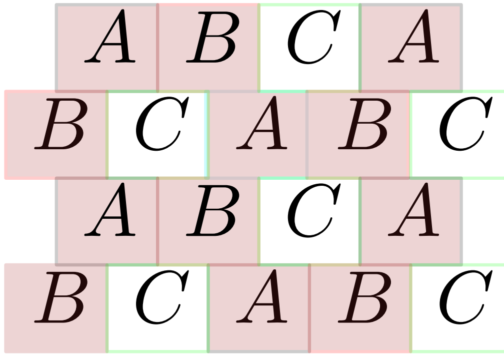

supported on correctable region $A$

$\Rightarrow PQ_U P^\dagger Q_U^\dagger \big|_{\mathcal{L}} \propto I_{\mathcal{L}}$ by definition of correctable regions

**Claim:** $U\big|_{\mathcal{L}}$ is an encoded Clifford group element

# Transversal gate $U$ & support of 'group commutator'

$P =$




Pauli

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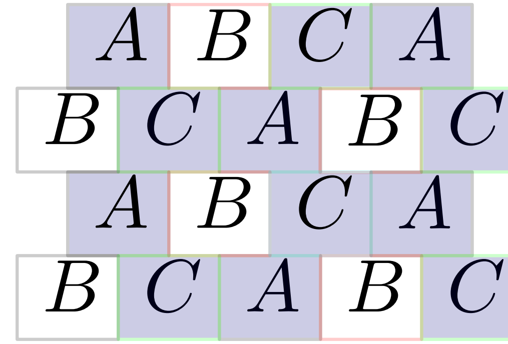
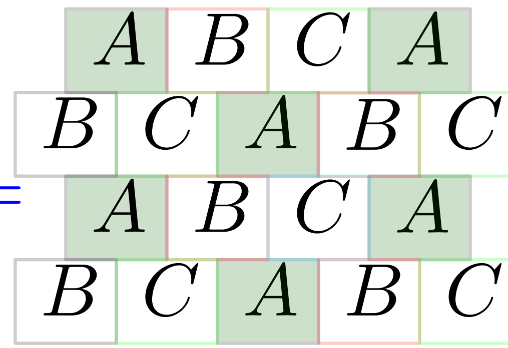



transversal

$PQ_UP^\dagger Q_U^\dagger =$



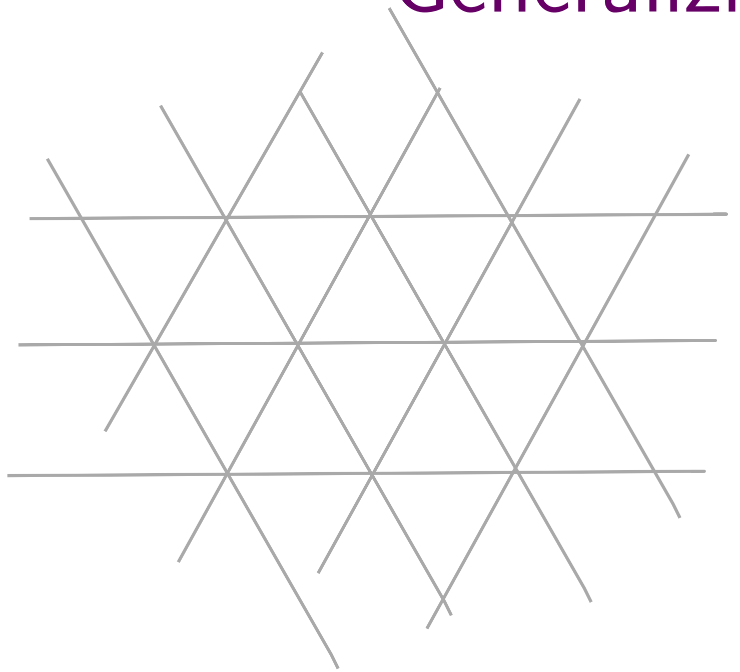

supported on correctable region $A$

$\Rightarrow PQ_UP^\dagger Q_U^\dagger\big|_{\mathcal{L}} \propto I_{\mathcal{L}}$ by definition of correctable regions

$\Rightarrow Q_UP\big|_{\mathcal{L}} = \pm PQ_U\big|_{\mathcal{L}}$ for all logical Pauli op $P, Q$

$\Rightarrow$ **Claim:** $U\big|_{\mathcal{L}}$ is an encoded Clifford group element

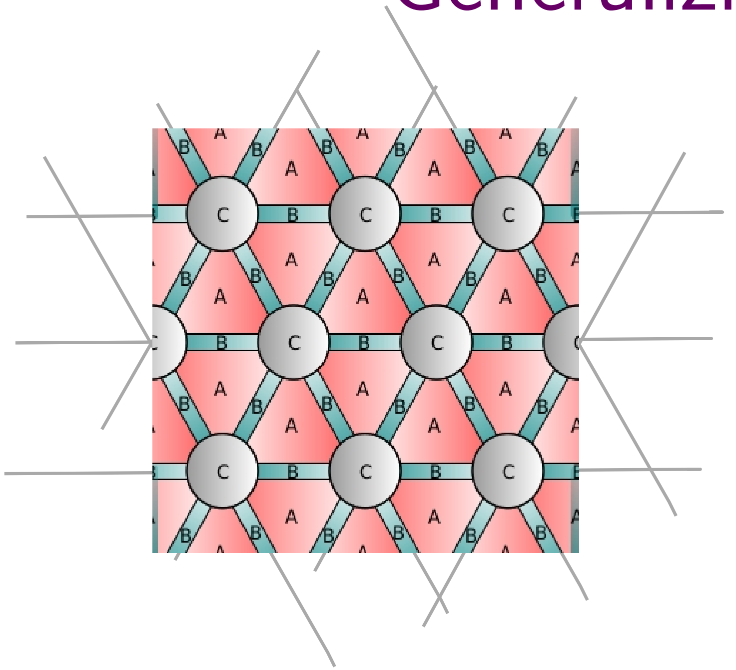
# Generalizing to higher dimensions



in $D = 2$

3 disjoint correctable regions $A, B, C$

# Generalizing to higher dimensions



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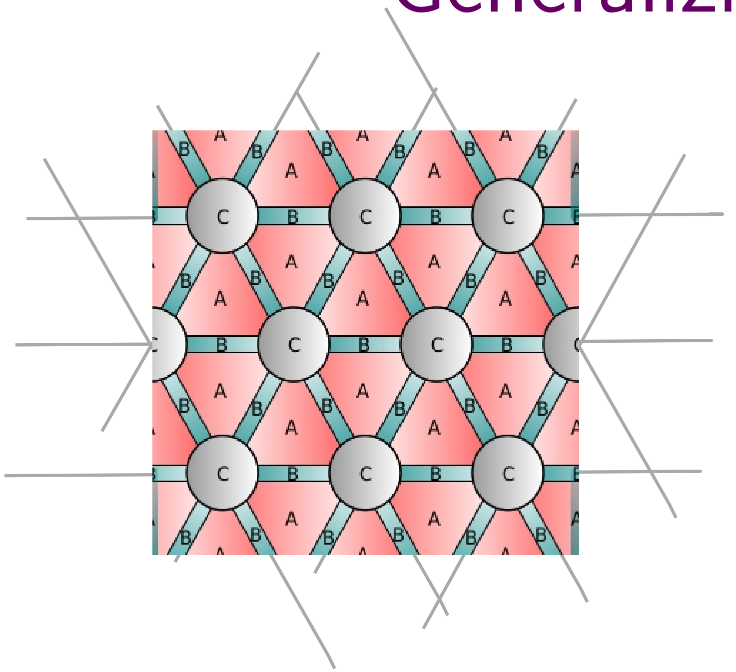
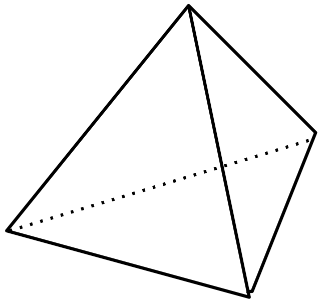
# Generalizing to higher dimensions




in $D = 2$

3 disjoint correctable regions $A, B, C$



in $D = 3$

4 disjoint correctable regions $A, B, C, D$

in $D$: $D + 1$ disjoint correctable regions

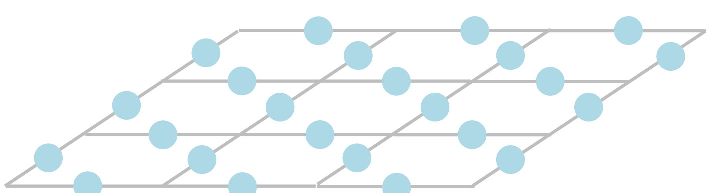
# The Clifford hierarchy and local stabilizer codes

Level $j+1$: $\mathcal{C}_{j+1} = \{\bar{U} \in \mathsf{U}(2^k) \mid \bar{U}\mathcal{C}_1\bar{U}^\dagger \subseteq \mathcal{C}_j\}$

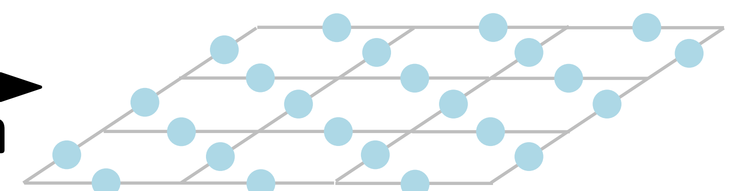
Pauli group ($\mathcal{C}_1$) — Level $j$ ($\mathcal{C}_j$)

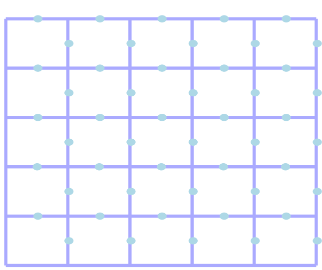
**Theorem:** For a $D$-dimensional local stabilizer code: $(D \geq 2)$

protected gates belong to $\mathcal{C}_D$.

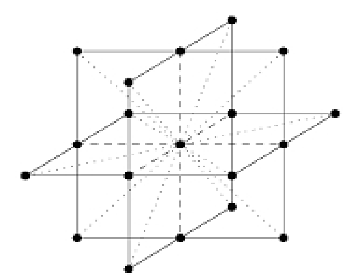


application of a **constant-depth circuit** $U$ →





Clifford group $\mathcal{C}_2$



$\mathcal{C}_3$

# Code deformation? no additional gates!

- Braiding of anyons?

Raussendorf, Harrington, PRL 98, 190504 (2007)
Fowler, Stephens Groszkowski, PRA 80, 052312 (2009)

**Theorem:** For a $D$-dimensional local stabilizer code: $(D \geq 2)$

protected gates belong to $\mathcal{C}_D$.

(Code deformation version) sequence of codes $\mathcal{L}^{(1)}, \ldots, \mathcal{L}^{(t)}$

$$\mathcal{L}^{(1)} \quad \rightarrow \quad \mathcal{L}^{(2)} \quad \rightarrow \quad \cdots \quad \rightarrow \quad \mathcal{L}^{(t)}$$

constant-depth circuit — constant-depth circuit — constant-depth circuit

overall logical operation belongs to $\mathcal{C}_D$

# Consequences for universality?

Level $j+1$: $\mathcal{C}_{j+1} = \{\bar{U} \in \mathsf{U}(2^k) \mid \bar{U}\mathcal{C}_1\bar{U}^\dagger \subseteq \mathcal{C}_j\}$

Pauli group ($\mathcal{C}_1$) — Level $j$ ($\mathcal{C}_j$)

**Theorem:** For a $D$-dimensional local stabilizer code: $(D \geq 2)$

protected gates belong to $\mathcal{C}_D$.

**Corollary:**

$2$-dimensional local stabilizer code

**Corollary:**

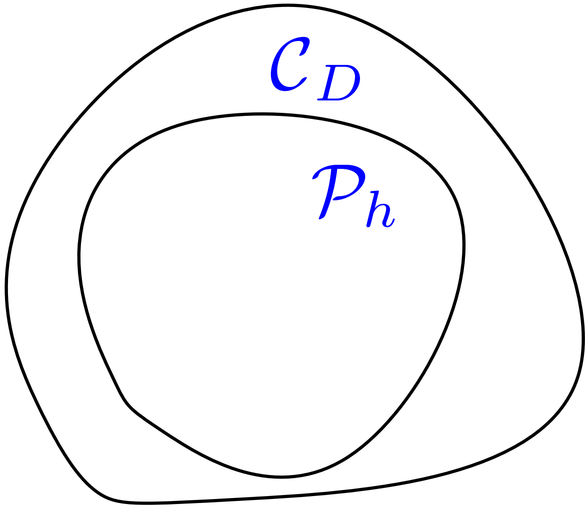
$\{\mathcal{L}_L\}_L$ family of $D$-dimensional local stabilizer codes such that $k = k(L)$ independent of $L$

} set of protected gates **not computationally universal**

# Proof of Corollary $h = const.$

Claim:

$\{\mathcal{L}_L\}_L$ family of $D$-dimensional local stabilizer codes such that $k = k(L)$ independent of $L$ } set of gates $\mathcal{P}_h$ implementable by depth-$h$ circuit generates group $\langle \mathcal{P}_h \rangle \subset \mathcal{C}_D$ ← finite



# Proof of Corollary $h = const.$

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Proof by contradiction:

(i) Suppose $\exists U_1, \ldots, U_m \in \mathcal{P}_h$ such that

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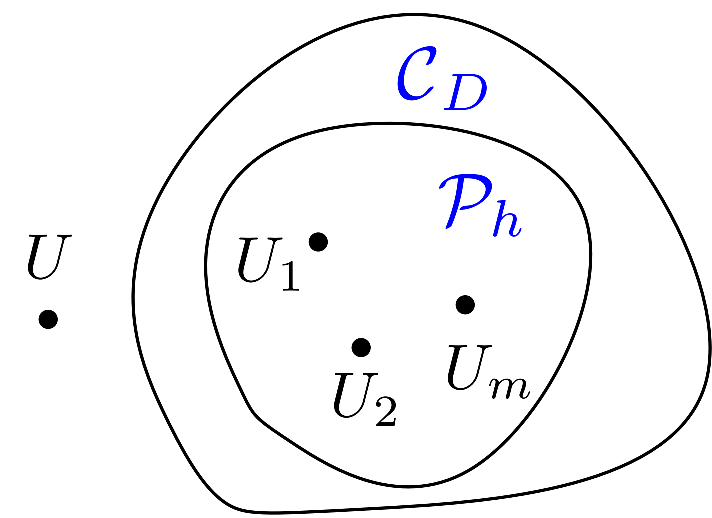

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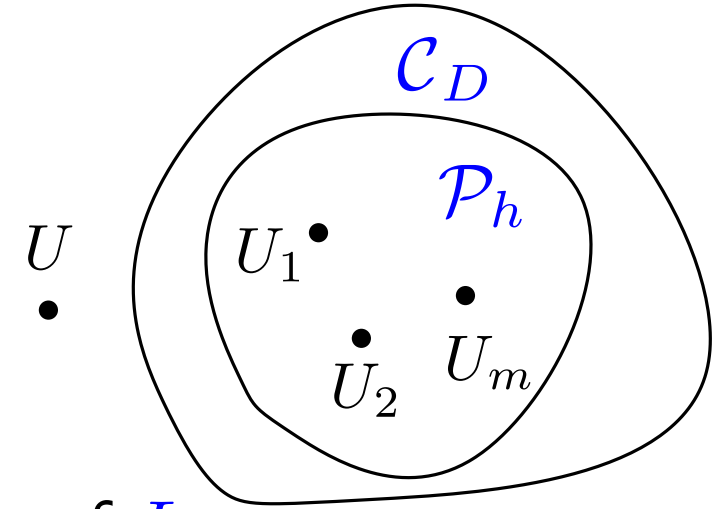


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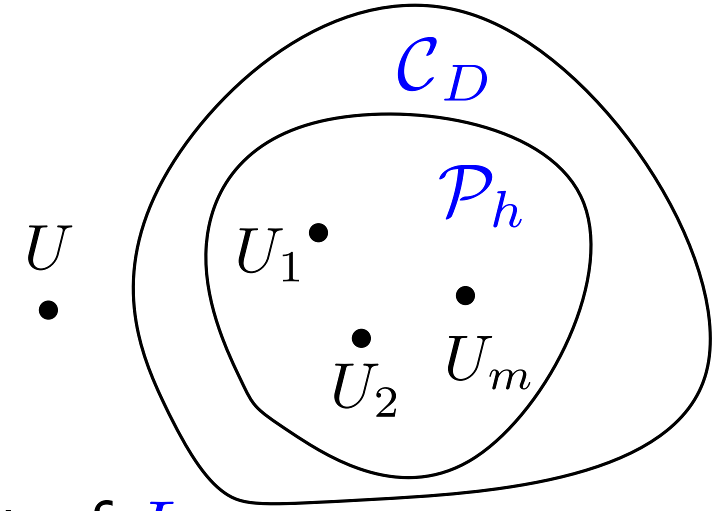


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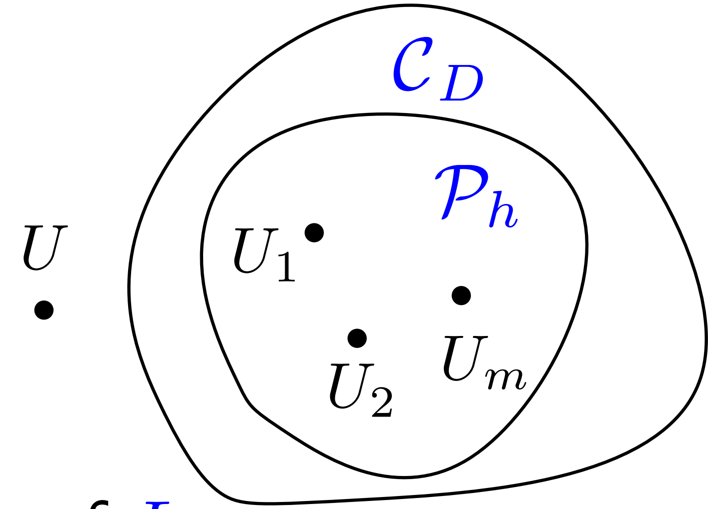


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# Alternatives to getting universality?

- 3D stabilizer codes with universal gate sets $k = k(L)$

  H. Bombin, M. A. Martin-Delgado: Topological Computation without braiding, PRL 98, 160502 (2007)

- magic state distillation

  Bravyi, Kitaev, Phys. Rev. A 71, 022316 (2005)
  Raussendorf, Harrington, Goyal NJP 9, 199 (2007)

- non-stabilizer codes

  Mochon, Phys. Rev. A 69, 032306 (2004)
  G. Brennen, M. Aguado, I. Cirac, New J. Phys. 11 053009 (2009)
  K, Kuperberg & Reichardt, Ann. Phys. 325, 2707 (2010)

Thank you for your attention!

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